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Fundamental Structures in Physics: A Categorical Approach†

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As already remarked by Eilenberg and MacLane in their seminal paper on category theory, preordered classes can be considered as thin categories, that is, categories for which each Hom-set contains at most one element. In this paper I briefly describe how this identification not only allows much of the theory of order structures to be reformulated in categorical terms, but also permits the application of general categorical techniques to specific order-theoretic problems.

1. INTRODUCTION

When one first encounters a new type of mathematical structure it is often useful to consider simple paradigm examples. In category theory, three such examples are preordered classes, namely categories with at most a single morphism between any two objects, monoids, namely categories with a single object, and Set, a category with an equilibrium between objects and morphisms. First, I shall give a brief expository discussion of the first of these, brief in the sense that proofs will be given only when they are short and instructive, and expository in the sense that I claim no originality in the examples I have chosen. Second, I shall indicate how this formalism not only allows an elegant reconstruction of much of the theory of order structures, but also has a direct application to the foundations of physics.

2. ORDER AND CATEGORY

In this section I shall summarize the category-theoretic presentation of order structures, following (Moore, 1995, 1997). First, I shall discuss the

† Dedicated to the memory of Fred Rüttimann, cruelly taken from us before his time. His mathematical insight and penetrating humour will be sorely missed.

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dual isomorphism between the categories JCLatt and MCLatt provided by the Galois adjunction, before specializing to the cases of atomistic and orthocomplemented lattices. Second, I shall consider closure operators in general and the equivalence between the categories JCALatt and CSpace in particular.

2.1. Category Theory

The basic notions of category theory are morphisms as relations between objects, functors as relations between morphisms, and natural transformations as relations between functors. For preordered classes we then have that:

- There exists a morphism $\alpha: a \to b$ if and only if $a < b$.
- $f: X \to Y$ is a functor if and only if it is isotone.
- There exists a natural transformation $\theta: f \to g$ if and only if $f < g$.

Further, the basic tools of category theory are limits as local universal constructions, adjoints as global universal constructions, and the adjoint functor theorem. For complete lattices we then have that:

- Products are meets and coproducts are joins.
- $f \dashv g$ if and only if $a_1 < g(a_2) \Leftrightarrow f(a_1) < a_2$
- $g(a_2) = \vee\{a_1 \in L_1 | f(a_1) < a_2\}$ and $f(a_1) = \wedge\{a_2 \in L_2 | a_1 < g(a_2)\}$

In particular, f preserves joins and g preserves meets. For notational convenience I shall often write $f \dashv f^*$ and $g_* \dashv g$. Note that $\mathrm{id} < g \circ f$ and $f \circ g < \mathrm{id}$, so that $f \circ g \circ f = f$ and $g \circ f \circ g = g$. First, $f(a) < f(a)$, so that $a < (g \circ f)(a)$ with associated inequalities $f < f \circ g \circ f$ and $g < g \circ f \circ g$. Second, $g(a) < g(a)$, so that $(f \circ g)(a) < a$ with associated inequalities $f \circ g \circ f < f$ and $g \circ f \circ g < g$.

2.2. Atoms and Orthocomplementations

If L is a lattice with minimal element 0, an element $p \neq 0$ is called an atom if $x < p$ implies $x = 0$ or $x = p$. We write Σ for the (possibly empty) set of atoms of L. The lattice L is then called atomistic if for each $a \in L$ we have that $a = \vee\{p \in \Sigma | p < a\}$. Now for $f \dashv g: L_2 \to L_1$ an adjunction between complete atomistic lattices, the following two conditions are equivalent:

- $f(\Sigma_1) \subseteq \Sigma_2 \cup \{0_2\}$.
- $(\forall p \in \Sigma_1)(\exists p_2 \in \Sigma_2) p_1 < g(p_2)$.

The Galois adjunction then restricts to an isomorphism between JCALatt and MCALatt$^{\mathrm{op}}$. On the other hand, an orthocomplementation is a map $': L \to L$ such that $a < b$ implies $b' < a'$, $a \wedge a' = 0$, $a'' = a$. Note that L then has maximal element $1 = 0'$, and $a \vee b = (a' \wedge b')'$. In particular, the

Galois adjunction defines isomorphisms between JCOLatt and $\text{JCOLatt}^{\text{op}}$ and MCOLatt and $\text{MCOLatt}^{\text{op}}$ via the following correspondences:

- $f^\dagger: L_2 \to L_1: a_2 \to f^*(a_2')'$.
- $g_\dagger: L_1 \to L_2: a_1 \to g_*(a_1')'$.

Indeed, if $f \dashv g$, then $f^\dagger \dashv g_\dagger$, since $a_1 < g_\dagger(a_2)$ if and only if $a_1 < f(a_2')'$ if and only if $f(a_2') < a_1'$ if and only if $a_2' < g(a_1')$ if and only if $g(a_1')' < a_2$ if and only if $f^\dagger(a_1) < a_2$. We do not require morphisms in JCOLatt or MCOLatt to preserve the orthocomplementation. Restricting either of these categories by this condition, we then obtain the category COLatt, for which $h_*(a_2') = h^*(a_2)'$ and $h^*(a_2') = h_*(g_2)'$, since $h_*(a_2') < a_1$ if and only if $a_2' < h(a_1)$ if and only if $f(a_1') = h(a_1)' < a_2$ if and only if $a_1' < f^*(a_2)$ if and only if $f^*(a_2)' < a_1$. In particular, $f^\dagger(a_2) = f^*(a_2')' = f_*(a_2'') = f_*(a_2)$ and $f_\dagger(a_2) = f^*(a_2')' = f^*(a_2'') = f^*(a_2)$.

2.3. Monads

A closure operator on the poset L is a map $T: L \to L$ such that: if $a < b$, then $T(a) < T(b)$; $a < T(a)$ for each $a \in L$; $(T \circ T)(a) < T(a)$ for each $a \in L$. In categorical terms, a closure operator is then an endofunctor T on L together with natural transformations η: id $\to T$ and $\mu: T \circ T \to T$. Imposing the obvious compatibility conditions $\mu \circ T\mu = \mu \circ \mu T$, $\mu \circ T\eta = \text{id}T$, and $\mu \circ \eta T = \text{id}T$, closure operators are then exactly monads. Now to each monad we can associate its Eilenberg–Moore cateory L^T, whose objects are pairs (a, α) for $\alpha: Ta \to a$ such that $\alpha \circ \eta_a = \text{id}_a$ and $\alpha \circ T(\alpha) = \alpha \circ \mu_a$, and whose morphisms are morphisms $f: a \to b$ such that $f \circ \alpha = \beta \circ T(f)$. In the context of posets, the Eilenberg–Moore category is just the set of fixed points of T with the induced order, since the existence of $\alpha: T(a) \to a$ is equivalent to the condition $T(a) < a < T(a)$. Further, for each monad on L we have the adjunction $F^T \dashv U^T$, where:

- $U^T: L^T \to L: (a, \alpha) \mapsto a; f \mapsto f$
- $F^T: L \to L^T: a \mapsto (T(a), \mu_a); f \mapsto T(f)$.

Since F^T preserves colimits and U^T preserves limits, the Eilenberg–Moore category associated to a closure operator T on the complete lattice L is itself a complete lattice, with $\wedge_T A = \wedge A$ and $\vee_T A = T(\vee A)$. Finally, any adjunction $M \dashv N: L_2 \to L_1$ induces the monad $(N \circ M, \eta, N\, \varepsilon\, M)$ on L_1, the unique functor $K: L_2 \to L_1^{N \circ M}$ satisfying $N = U \circ K$ and $F = K \circ M$ being given by $K: a \mapsto (Na, N\, \varepsilon_a)$; $f \mapsto Nf$. In particular, the Eilenberg–Moore category associated to $N \circ M$ is just the image of N.

2.4. Closure Spaces

A closure operator T on the atomistic lattice L is called simple if $T(0) = 0$ and $T(\Sigma_L) \subseteq \Sigma_L$, where Σ_L is the set of atoms of L. Then L^T is atomistic and has the same atoms as L. In particular, let us call a closure space a set Σ together with a simple closure operator T on $\mathcal{P}(\Sigma)$. Then to each closure space we can associate a complete atomistic lattice. The converse is also true, since for each complete atomistic lattice we can define the adjunction $\pi_L \dashv i_L$ by

- i_L: $L \to \mathcal{P}(\Sigma_L)$: $a \mapsto \{p \in \Sigma_L | p < a\}$
- π_L: $\mathcal{P}(\Sigma_L) \to L$: $A \mapsto \vee A$

with corresponding closure operator $i_L \circ \pi_L$: $\mathcal{P}(\Sigma_L) \to \mathcal{P}(\Sigma_L)$: $A \mapsto \{p \in \Sigma_L | p < \vee A\}$. Let us define a morphism in CSpace to be a partially defined map θ: $\Sigma_1 \backslash K_1 \to \Sigma_2$ satisfying equivalently $\theta(T_1 A_1 \backslash K_1) \subseteq T_2\theta(A_1 \backslash K_1)$ or $T_1(K_1 \cup \theta^{-1}(T_2 A_2)) = K_1 \cup \theta^{-1}(T_2 A_2)$. The above object correspondence then extends to the categorical equivalence between the JCALatt and CSpace defined by the adjunction $\mathbf{L} \dashv \mathbf{C}$, where

- $\mathbf{L}$: $(\Sigma, T) \mapsto \mathcal{P}(\Sigma)^T$; $\theta \mapsto f_\theta$: $\mathcal{P}(\Sigma_1)^{T_1} \to \mathcal{P}(\Sigma_2)^{T_2}$: $A_1 \mapsto T_2\theta(A_1 \backslash K_1)$
- $\mathbf{C}$: $L \mapsto (\Sigma_L, i_x \circ \pi_x)$; $f \mapsto \theta_f$: $\Sigma_1 \backslash f^*(0_2) \to \Sigma_2$: $p_1 \mapsto f(p_1)$

Finally, this equivalence restricts to an equivalence between JCAOLatt and OSpace, the category of sets equipped with an orthogonality relation. Indeed, $A \mapsto A^{\perp\perp}$ is a simple closure operator on $\mathcal{P}(\Sigma)$, and A is biorthogonal if and only if $A = \{p \in \Sigma | p < \vee A\}$.

3. APPLICATIONS

In this section I shall summarize some applications of adjunctions in the foundations of physics. First, I shall discuss the primitive notions of state-property duality and classical variables following Moore (1999) and Piron (1990, §1). Second, I shall consider the derived notions of evolutions and observables, following Faure *et al.* (1995) and Piron (1976, §2), respectively.

3.1. State Property Duality

A physical system may be described by either its properties, the elements of reality associated to definite experimental projects that one could effectuate on the system, or by its states, construed as abstract names for singular realizations of the physical system. The set of properties is then a complete lattice, the meet being obtained from the product definite experimental project, whereas the set of states is equipped with the orthogonality relation defined by calling two states orthogonal if they can be distinguished by a given definite experimental project. The standard axioms then imply that the set of properties is a complete atomistic ortholattice and that the set of states is an orthogonality space. In particular, the property and state representations are dual via the equivalence between JCALatt and OSpace, states corresponding to the set of their actual properties and properties corresponding to the set of states for which they are actual. Further, the adjunction condition has a direct physical interpretation, since if $f \dashv g$, then $g(a_2)$ is the weakest property in L_1 guaranteeing the actuality of a_2 in L_2, whereas $f(a_1)$ is the strongest property in L_2 whose actuality is guaranteed by $a_1 \in L_1$. Note that the application of this duality to the Hilbertian context leads to both the characterization of semilinear maps as representations for morphisms of projective geometries and the inner product as a realization of the canonical polarity between a projective geometry and its dual.

3.2. Classical Variables

An element c of the lattice L is called central if L admits a direct product decomposition $\pi: L_1 \times L_2 \to L$ with $c = \pi(1, 0)$; in which case c has unique complement $c' = \pi(0, 1)$. In particular, the center Z of any ortholattice is a Boolean subalgebra. Now for any complete lattice we have that $(\vee A) \wedge c = \vee(A \wedge c)$ for any $A \subseteq L$. If Z is complete and $(\vee C) \wedge a = \vee(C \wedge a)$ for any $C \subseteq Z$, then L is called a $\mathscr{Z}$-lattice. Note that this is the case for atomistic ortholattices, in which case Z is also atomistic with atoms $\alpha \in \Omega$ induced by the left adjoint to the inclusion $i: Z \to L$. We then obtain the central decomposition, since $a = 1 \wedge a = (\vee\Omega) \wedge a = \vee(\Omega \wedge a)$. Physically, the central elements of a property lattice coincide with classical properties, that is, a property c such that for each state either c or c' is actual. Indeed, let c be classical. Then for each atom $p \in L$ either $p < c$ or $p < c'$, so that for each $p < a$ either $p < a \wedge c$ or $p < a \wedge c'$, and $a = \vee\{p | p < a\} = (\vee\{p | p < a \wedge c\}) \vee (\vee\{p | p < a \wedge c'\}) = (a \wedge c) \vee (a \wedge c')$. Hence c is central. On the other hand, let $c \in L$ be central, and $p \in L$ be an atom. Then since $p \wedge c < p$, either $p \wedge c = p$ or $p \wedge c = 0$. In the first case $p < c$. In the second case $p = (p \wedge c) \vee (p \wedge c') = 0 \vee (p \wedge c') = p \wedge c'$, and so $p < c'$. Hence c is classical.

3.3. Observables

An observable is an orthomorphism $\mu: \mathbf{B} \to L$, where $\mathbf{B}$ is a complete Boolean algebra representing the measurement scale as a directed limit of refinements of possible outcome sets. For $\mathbf{E}$ the set of atoms of $\mathbf{B}$ we then obtain the spectrum:

- $N = \mu^*(0)$
- $D = \vee\{E \in \mathbf{E} | \; E < N'\}$
- $C = D' \wedge N'$

Here N represents the kernel, since $\mu(B) = 0$ if and only if $\mu(B) < 0$ if and only if $B < \mu^*(0)$. Further, D represents the discrete spectrum, since by construction $[0, D]$ is atomistic and so a power set. Finally, C represents the continuous spectrum, since by construction $[0, C]$ is atomless. Note that observables can be trivially decomposed into classical components. In particular, classical observables have discrete spectrum and so are determined by functions $f: \Sigma \to \mathbf{E}$. Explicitly, the observable $\mu: \mathbf{B} \to \mathscr{L}$ preserves the meet and so has left adjoint $\mu_*: \mathscr{L} \to \mathbf{B}: a \mapsto \wedge\{B \in \mathbf{B} \mid a < \mu(B)\}$. Defining $E_p = \mu_*(p)$, one can prove that the E_p exhaust all atoms $E < D$ and that $C = 0$. We can then define $f: \Sigma \to \mathbf{E}: p \mapsto E_p$. On the other hand, for the case of a system described by a separable Hilbert space $\mathscr{H}$, Wade's theorem implies that each observable is generated by a self-adjoint operator, since the image of μ is the projection lattice of a Boolean von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$.

3.4. Evolutions

An externally imposed evolution can be partially described by its interpretation as an initial segment of experimental projects. Explicitly, to each experimental project α_2 defined at time t_2 one can associate the experimental project $\alpha_1 = \Phi(\alpha_2)$ defined at time t_1 by the prescription, "Evolve the system from time t_1 and time t_2 and effectuate α_2." Since Φ preserves products, it then induces a map $\varphi: L_2 \to L_1$ which preserves nonempty meets, and so has a left adjoint $\psi: [0, \varphi(1_2)] \to L_2$. Note that in general the domain $[0, \varphi(1_2)]$ is orthocomplemented if and only if L_1 is orthomodular, in which case we can extend the domain of ψ to all of L_1 by composition with the appropriate Sasaki projection. Now for p_1 an atom, $\psi(p_1)$ is the strongest final property which is actual by the evolution. In the simplest case, this property will be either an atom, thereby determining the final state, or 0_2, indicating that the initial state could be destroyed by the evolution. Further, if the evolution is sufficiently stable, two orthogonal final states must arise from two orthogonal initial states, since if α_2 separates the final states, then $\Phi(\alpha_2)$ separates the initial states. In the Hilbertian context we can then recover the description of maximal deterministic evolutions by unitary flows. On the other hand from the definition of ideal measurements of the first kind as consistent experiments leading to minimal perturbation, we recover both their characterization as projections and the usual formula for the *a priori* probability.

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